Symplectic Geometry

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– Symplectic Bundles –

Definition 1. A **symplectic vector space** is a real vector space V with a skew-symmetric non-degenerate bilinear map $\omega : V \times V \to \mathbb{R}$, called the **symplectic form**.

We may represent ω by the matrix $\Omega = (\omega_{ij})$ so that

$$\omega(v, w) = \omega_{ij} v^i w^j = v^T \Omega w.$$

By definition $\omega_{ij} = -\omega_{ji}$ and $\det \Omega \neq 0$.

Proposition 2. *Any symplectic vector space is even-dimensional, and there exists a basis* $\{e_1, \ldots, e_{2n}\}$ *such that*

$$\Omega = \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & -1 & 0 \end{pmatrix}.$$

Proof. Let $e_1 \neq 0$ be any vector. Since ω is non-degenerate, there exists a vector v for which $\omega(e_1, v) \neq 0$, and we take $e_2 = v/\omega(e_1, v)$. Now V decomposes as $W \oplus W^\perp$, where W is the subspace spanned by e_1 and e_2, and

$$W^\perp = \{v \in V : \omega(v, w) = 0 \text{ for all } w \in W\}.$$

This splitting can be obtained from the projection

$$\pi_W : V \to W, \quad v \mapsto -\omega(e_2, v)e_1 + \omega(e_1, v)e_2.$$

According to this splitting the matrix Ω has the form

$$\Omega = \begin{pmatrix} 0 & 1 & \\ -1 & 0 & \\ & & \Omega' \end{pmatrix},$$

where Ω' denotes the restriction of Ω to $W^\perp$, and we continue by induction. □

Definition 3. A **symplectic bundle** is a real vector bundle $\pi : E \to M$ with a smooth section ω of $\bigwedge^2 E^*$ (the symplectic form) such that (E_x, ω_x) is a symplectic vector space for all $x \in M$.

There is a close relation between complex Hermitian spaces and real symplectic spaces. Let $E \to M$ be an n-dimensional complex bundle with a Hermitian form $h(-,-)$. Let L be the realification of E. Then we can write

$$h(u, v) = g(u, v) + i\omega(u, v),$$

where g and ω are real bilinear forms on L. Since $h(v,u) = \overline{h(u,v)}$, the form $g(u,v)$ is symmetric, while $\omega(u,v)$ is skew-symmetric, defining a Riemannian metric and symplectic form on L, respectively. Moreover, we have a complex structure J on L given by multiplication by i on E. Note that

$$ig(u,v) - \omega(u,v) = i\,h(u,v) = h(u,iv) = g(u,Jv) + i\omega(u,Jv),$$

from which follows that

$$g(u,v) = \omega(u,Jv).$$

Definition 4. A complex structure J is **positive** if the bilinear form $\omega(u,Jv)$ is symmetric and positive definite (so that $g(u,v) := \omega(u,Jv)$ defines a metric).

Proposition 5. *For any symplectic bundle L, there exists a positive complex structure $J \in \mathrm{Hom}(L,L)$. Any two such structures are homotopic.*

Proof. Let $A : L \to L$ be given by $g(u,v) = \omega(u,Av)$. Then $g(u,A^{-1}v) = \omega(u,v) = -\omega(v,u) = -g(A^{-1}u,v)$, and hence

$$g(u,A^{-2}v) = -g(A^{-1}u,A^{-1}v) = g(A^{-2}u,v),$$

which shows that A^{-2} is self-adjoint w.r.t. g and negative-definite. Let $B = (-A^{-2})^{1/2}$ and set $J = AB$. Since A and B commute, we have

$$J^2 = A^2B^2 = -A^{-2}A^2 = -1,$$

so J is a complex structure. It is positive because

$$\omega(u,Jv) = \omega(u,ABv) = g(u,Bv) > 0$$

since B is positive-definite w.r.t. g. Any two positive complex structures are homotopic via a homotopy for the Riemannian metrics. □

Corollary 6. *Any symplectic bundle is the realification of a Hermitian bundle. Any two positive complex structures are isomorphic.*

– Symplectic Manifolds –

Definition 7. A **symplectic manifold** is a manifold M with a closed non-degenerate 2-form ω on M, called the **symplectic form**.

Example 8. The standard example is $M = \mathbb{R}^{2n}$ with coordinates $q^1,\dots,q^n,p_1,\dots,p_n$ and

$$\omega = \sum_{i=1}^{n} dq^i \wedge dp_i.$$

The second standard model is $M = \mathbb{C}^n$ with coordinates $z_1,\dots,z_n$ and

$$\omega = -i\sum_{i=1}^{n} d\overline{z} \wedge dz.$$

Any symplectic form ω defines a canonical orientation and volume element ω^n.

Example 9. On a compact symplectic manifold ω cannot be exact. Namely, if $\omega = d\lambda$, then

$$\int_M \omega^n = \int_M d(\lambda \wedge \omega^{n-1}) = 0,$$

but the left-hand side cannot be zero since ω^n is an orientation form.

The form ω defines a bundle isomorphism

$$\omega^\flat : TM \to T^*M, \quad X \mapsto \omega(X, -),$$

identifying vector fields and one-forms. The vector fields corresponding to exact one-forms are called **Hamiltonian vector fields**. The vector fields corresponding to closed one-forms are called **locally Hamiltonian**. Notation-wise, for a function $H \in C^\infty(M)$ we have

$$dH = \omega(X_H, -).$$

A symplectic form yields a Poisson algebra structure on $C^\infty(M)$, given by

$$\{f, g\} = \omega(X_f, X_g) = X_f(g).$$

Clearly the bracket is anti-symmetric, and $\{f, -\} = X_f$ is derivation. The Jacobi-identity follows from $d\omega = 0$.

Definition 10. A **symplectomorphism** is a diffeomorphism $f : M \to M$ with $f^*\omega = \omega$.

Proposition 11. *The flow of a Hamiltonian vector field (time-dependent in general) is a symplectomorphism.*

Proof. Let $X(t)$ be a (time-dependent) Hamiltonian vector field, i.e. $\omega(X(t), -) = dH(t)$, and let f_t be its flow, i.e. $\dot{f}_t(x) = X(f_t(x), t)$. Then

$$\frac{d}{dt} f_t^*\omega = f_t^* \mathcal{L}_{X(t)}\omega = f_t^*(d\iota_{X(t)}\omega + \iota_{X(t)} d\omega) = 0,$$

since $d\omega = 0$ and $\iota_{X(t)}\omega = dH(t)$. □

Proposition 12. *Any smooth family f_t of symplectomorphisms is generated by a locally Hamiltonian vector field.*

Proof. Define the generating vector field

$$X(t)u = (f_t^{-1})^* \frac{d}{dt} f_t^* u.$$

This vector field is indeed locally Hamiltonian as

$$0 = \frac{d}{dt} f_t^*\omega = f_t^* \mathcal{L}_{X(t)}\omega,$$

so that $\mathcal{L}_{X(t)}\omega = d(\omega(X(t), -)) = 0$. □

Example 13 (Classical mechanics)**.** Let $Q \simeq \mathbb{R}^n$ be an n-dimensional smooth manifold (*configuration space*), and $M = T^*Q \simeq \mathbb{R}^{2n}$ with standard coordinates $q^1, \dots, q^n, p_1, \dots, p_n$. The *tautological* 1*-form* or *Louiville* 1*-form* is given by

$$\theta = \sum_{i=1}^{n} p_i dq^i,$$

and the corresponding canonical symplectic form is

$$\omega = -d\theta = \sum_{i=1}^{n} dq^i \wedge dp_i.$$

The vector field corresponding to a *Hamiltonian function* is

$$X_H = \left(\frac{\partial H}{\partial p_i}, -\frac{\partial H}{\partial q^i} \right),$$

from which we recognize *Hamilton's equations.*

Definition 14. Let H be a Hamiltonian function. An **integral of motion** (w.r.t. H) is a function f with $\{H, f\} = X_H(f) = 0$. In particular, it is constant on any trajectory generated by X_H. Note that H itself is a integral of motion (conservation of energy). Note that the integrals of motion form a sub-Poisson algebra of $C^\infty(M)$.

– The Darboux Theorem –

Theorem 15 (Darboux's theorem)**.** *Let (M, ω) be a symplectic manifold. Then for any point $x \in M$, there exist local coordinates $q^1, \dots, q^n, p_1, \dots, p_n$ such that ω is given by $\sum_{i=1}^n dq^i \wedge dp_i$.*

Proof. Locally around x, we can write $\omega = \frac{1}{2}\omega_{ij} dx^i \wedge dx^j$. Consider $\omega_0 = \frac{1}{2}\omega_{ij}(x) dx^i \wedge dx^1$, which is also a symplectic form in the same neighborhood of x. Moreover, for a sufficiently small neighborhood around x, we have a family of symplectic forms

$$\omega(t) = (1-t)\omega_0 + t\omega, \quad t \in [0, 1].$$

Since $\omega - \omega_0$ is closed, using Poincaré's lemma we can locally around x write $\dot{\omega}(t) = \omega - \omega_0 = -d\lambda$ for some 1-form λ. Since $\omega - \omega_0$ vanishes at x, we may choose λ to vanish at x up to second order. Now let $X(t)$ be the vector field defined by $\iota_{X(t)}\omega(t) = \lambda$, and let φ_t be the flow of $X(t)$ around x_0. Since $\varphi_t(x) = x$ for all $t \in [0, 1]$, the flow φ_t exists on the whole interval $t \in [0, 1]$ sufficiently close to x. Now,

$$\frac{d}{dt}\varphi_t^*\omega(t) = \varphi_t^* \left(\frac{\partial \omega(t)}{\partial t} + \mathcal{L}_{X(t)}\omega(t) \right) = \varphi_t^* \left(\frac{\partial \omega(t)}{\partial t} + d(\iota_{X(t)}\omega(t)) \right) = \varphi_t^* \left(-d\lambda + d\lambda \right) = 0,$$

using the Cartan formula. This implies that $\varphi_1^*\omega = \varphi_1^*\omega(1) = \varphi_0^*\omega(0) = \omega_0$, so φ_1 is the desired diffeomorphism. Finally, by a linear change of variables the form ω_0 can be reduced to the canonical form. □

Theorem 16 (More general Darboux)**.** *Let N be a compact submanifold of a manifold M and let ω_0, ω_1 be two symplectic forms on a neighborhood of N. If $\omega_0|_N = \omega_1|_N$, then there exist two neighborhoods U, V of N and a diffeomorphism $f : U \to V$ such that f and df are the identity on N, and $f^*\omega_1 = \omega_0$.*

– The Generating Function –

Proposition 17. *Let x^i and y^i be coordinates on $\mathbb{R}^{2n}$, such that $y = f(x)$ for some symplectomorphism $f : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$. Set $z = \frac{x+y}{2}$ and assume that this defines x as an implicit function of z. Then there exists a function $S(z)$ such that*

$$dS(z) = \omega\left(\frac{y-x}{2}, dz\right).$$

Proof. It suffices to check that the right-hand side is closed, and indeed

$$d\left(\omega\left(\frac{y-x}{2}, dz\right)\right) = \frac{1}{2}\omega(dy-dx, dz) = \frac{1}{4}\left(\underbrace{\omega(dy,dy)-\omega(dx,dx)}_{=0,\text{ as } f \text{ is symplectomorphism}} + \underbrace{\omega(dy,dx)-\omega(dx,dy)}_{=0}\right) = 0.$$

□

The function $S(z)$ is called a *(symmetrized) generating function* of the symplectomorphism. If we know it, we can reconstruct f. Namely,

$$x^i = z^i + \omega^{ij}\frac{\partial S(z)}{\partial z^j},$$
$$y^i = z^i - \omega^{ij}\frac{\partial S(z)}{\partial z^j}.$$

If the first defines z as an implicit function of x, we substitute this in the second equation to get the map f. By the same argument as above we find that this f is again a symplectomorphism.

Proposition 18. *Any symplectomorphism is locally homotopic to the identity.*

– Symplectic Connections –

Definition 19. Let $E \to M$ be a symplectic bundle. A **symplectic connection** is a connection ∇ on E preserving the symplectic form ω, that is

$$(\nabla_X \omega)(u,v) = X(w(u,v)) - \omega(\nabla_X u, v) - \omega(u, \nabla_X v) = 0$$

for any sections $u, v \in \Gamma(E)$ and vector field X.

A symplectic connection on a symplectic manifold (M, ω) is a torsion-free affine connection preserving the symplectic form.

Proposition 20. *There exists a symplectic connection on any symplectic manifold.*

Proof. Let ∇ be any torsion-free connection (e.g. the Levi-Civita connection), and try a connection of the form $\nabla' = \nabla + \delta\Gamma$, with $\delta\Gamma = \delta\Gamma^i{}_{jk}$ a tensor field symmetric in jk. Then from

$$Z(\omega(X,Y)) = (\nabla_Z\omega)(X,Y) + \omega(\nabla_Z X, Y) + \omega(X, \nabla_Z Y)$$

and a similar expression for ∇', we obtain

$$(\nabla'\omega)(X,Y) = (\nabla\omega)(X,Y) - \omega(\delta\Gamma(Z,X),Y) - \omega(X, \delta\Gamma(Z,Y)),$$

which we want to be zero, that is

$$\nabla_k \omega_{ij} = \omega_{\ell j} \delta \Gamma^{\ell}{}_{ik} + \omega_{i\ell} \delta \Gamma^{\ell}{}_{jk} = \delta \Gamma_{ijk} - \delta \Gamma_{jik}. \tag{1}$$

We impose an additional condition on $\delta\Gamma$ that $\delta\Gamma_{(ijk)} = 0$. Then

$$\delta\Gamma_{ijk} = \delta\Gamma_{ijk} - \frac{1}{3}\left(\delta\Gamma_{ijk} + \delta\Gamma_{jki} + \delta\Gamma_{kij}\right) = \frac{1}{3}\left(\delta\Gamma_{ijk} - \delta\Gamma_{jik}\right) + \frac{1}{3}\left(\delta\Gamma_{ikj} - \delta\Gamma_{kij}\right) = \frac{1}{3}\nabla_k \omega_{ij} + \frac{1}{3}\nabla_j \omega_{ik}$$

does the trick. It satisfies (1) since

$$\delta\Gamma_{ijk} - \delta\Gamma_{jik} = \frac{1}{3}\left(\nabla_k \omega_{ij} + \nabla_j \omega_{ik} - \nabla_k \omega_{ji} - \nabla_i \omega_{jk}\right) = \nabla_k \omega_{ij} + \frac{1}{3}\left(\nabla_j \omega_{ik} + \nabla_i \omega_{kj} + \nabla_k \omega_{ji}\right) = \nabla_k \omega_{ij},$$

using that $d\omega = 0$ and the fact that ∇ is torsion-free. □

Remark 21. From the homogeneous equation corresponding to (1), we see that any two symplectic connections differ by a completely symmetric tensor $\delta\Gamma_{ijk}$.

Theorem 22. *At any point x_0, there exists a local Darboux coordinate system centered at x_0 such that $\Gamma_{ijk}(0) = 0$ and*

$$\Gamma_{ijk}(x) x^i x^j x^k$$

vanishes at $x = 0$ up to infinite order. Two such systems differ up to infinite order by a linear symplectic change of variables.

– Kirillov Form on Coadjoint Orbits –

Example 23. Let G be a Lie group and $\mathfrak{g}$ its corresponding Lie algebra. Any coadjoint orbit $\mathcal{O}_\mu = \{\mathrm{Ad}_g^* \mu : g \in G\}$ can be described via

$$\begin{aligned} G/G_\mu &\xrightarrow{\sim} \mathcal{O}_\mu \\ g &\mapsto \mathrm{Ad}_g^* \mu. \end{aligned}$$

There is a symplectic structure on $\mathcal{O}_\mu$ given by the Kirillov form

$$\omega_\nu(\mathrm{ad}_X^* \nu, \mathrm{ad}_Y^* \nu) = \nu([X, Y]).$$

Note that there is an isomorphism

$$\begin{aligned} \mathfrak{g}/\mathfrak{g}_\nu &\xrightarrow{\sim} T_\nu \mathcal{O}_\mu \\ X &\mapsto \mathrm{ad}_X^* \nu = \nu([X, -]), \end{aligned}$$

which shows both that ω is well-defined, and that ω is non-degenerate. To show that ω is closed, pullback ω to G, and show it is closed (even exact) there. Use the Maurer–Cartan form.